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Free-surface flow over a step

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A transformation technique is used to solve the problem of steady free-surface flow of an ideal fluid over a semi-infinite step in the bottom. Application of the exact free-surface condition results in a nonlinear integro-differential equation for the free-surface angle and solutions of this equation are dependent on step height and Froude number. Linearized solutions, based upon small step height are presented and indicate that the nature of the free surface formed depends on whether the upstream flow is subcritical or supercritical. As the step height is increased, solutions to the exact nonlinear equations are obtained using the predictions of the linear theory, or possibly a previous nonlinear solution, as an initial estimate.

1. Introduction

This paper considers the steady free-surface flow of a stream of ideal fluid which is obstructed by a semi-infinite step on the bottom; the restoring force is gravity. This problem finds application in hydraulic and coastal engineering and is also expected to provide a qualitative description of the flow caused by a long body moving close to the sea bed.

Free-surface flows over various obstacles have been studied for at least the last century. Kelvin (1886) considered the stationary wave pattern caused by finite elevations or depressions in the bed of a stream and also developed expressions for the hydrodynamic forces acting on these obstacles. Several authors have used concentrated singularities to model both finite and infinite bodies which are moving in streams of finite or infinite depth, e.g. Havelock (1927) and Gazdar (1973). The motion of hydrofoils on or within the surface of a fluid has been studied under the assumptions of either potential flow or free-streamline flow, e.g. Cumberbatch (1958), Moiseev & Ter-Krikorov (1958) and Squire (1957). Reviews of these works, which are all mathematically similar and made tractable by the linearization of the exact free-surface condition, are given in Wehausen & Laitone (1980) and Seeger & Temple (1965).

Approximate nonlinear theories for purely periodic wave motions were first developed by Stokes (1847), by using an expansion in amplitude. Later Korteweg & de Vries (1895) produced a third-order theory for long waves in shallow water. This cnoidal wave theory was examined by Benjamin & Lighthill (1954) in a new light and, for convenience, their work will be referred to as the BL theory for the remainder

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of this paper. It was shown that the volume flow per unit span Q, the total head R, and the rate of flow of horizontal momentum (corrected for pressure force, and divided by density) S, determined the wavetrain uniquely in cnoidal theory. It was further suggested that these three quantities probably determined the wavetrain uniquely in general. To this end, the combinations of Q, R and S that represent possible flows in the general case were illustrated by a diagram which showed the area of parameter space occupied by wavelike solutions. However, at that time the boundary corresponding to what was thought to be the highest wave was not known. It was shown that the diagram could be used in steady flow problems to determine what losses of momentum and energy a given stream could undergo and this determines the maximum wave resistance on an obstacle spanning a subcritical stream. Detailed comparisons of the present results with their work support the general applicability of the theory.

Various types of obstructed free-surface flow have been analysed using approximate nonlinear theories of similar accuracy to the cnoidal theory of Benjamin & Lighthill, e.g. a sluice gate projecting into the free surface by Benjamin (1956) who also considered the effect of arbitrary bumps on the bottom of a stream (Benjamin 1970). A similar problem was also examined by Miles (1986).

A theory of directed fluid sheets was developed by Green & Naghdi (1976). It was shown that in the absence of dissipation the equation that determines the free surface reduced to the corresponding equation of Benjamin & Lighthill involving Q, R and S. Allowing for the possibility of dissipation, Naghdi & Vonsarnpigoon have considered the steady subcritical flow past a step (1986*a*) and also the effect of bumps on the bottom of a stream (Naghdi & Vonsarnpigoon (1986*b*). In the case of a step on the bottom, the conditions that were applied across the step essentially determined the drag on the step. In the absence of dissipation, the results are compared with the present analysis and large discrepancies appear, the reasons for which are discussed fully.

The advent of high-speed digital computers appears to have stimulated the solution of problems in which the exact, nonlinear, free-surface condition is retained. As the position of the free surface is, *a priori*, unknown there have been a variety of approaches to the solution of this type of problem. Von Kerczek & Salvesen (1977) considered the generation of waves on the surface of a stream by an imposed distribution of pressure on the free surface. By expressing all field equations in finite-difference form and assuming a solution of initially linear form they were able to iterate both the mesh points used and the solution at these mesh points until the equations of motion, boundary conditions and exact free-surface condition were satisfied. Haussling & Coleman (1977) developed a boundary-fitted coordinate system which numerically generates a new coordinate system in which coordinate lines in the new system correspond to physical boundaries. This technique can be used for both steady and unsteady free-surface flows. Shanks & Thompson (1977) have extended this type of method to enable viscous, time-dependent flows to be analysed.

The use of velocity potential ϕ, and stream function ψ, as independent variables was first introduced by Stokes (1880) and has been successfully used by many authors to solve problems concerned with solitary waves or purely periodic wave motions. A recent example of this approach can be found in Cokelet (1977) where steep gravity waves in water of finite depth are analysed numerically using a (ϕ, ψ)-formulation and the exact free-surface condition. This work gives an extensive tabulation of purely periodic nonlinear wave parameters and supplements the BL theory by showing that the barrier that closes the region in (RS)-space in which steady waves can exist is formed by waves with the greatest total head and momentum flux but does not exactly correspond to waves of greatest height.

An analysis of an obstructed free-surface flow in which the exact free-surface condition was retained is given by Forbes & Schwartz (1982) who consider flow over a semicircular bump in the bed of a stream of finite depth. These authors used a Joukowski transformation to remove the stagnation point at the front of the obstruction and reformulated the problem in a transform plane by use of a (ϕ, ψ)-coordinate system to obtain an integro–differential equation for the transformed free-surface elevation. The free-surface elevation in the physical plane together with other quantities of interest, e.g. the drag force on the semicircle, were then calculated from the solution to the integro–differential equations and appropriate inversion and integration of the Joukowski transformation. The analytic linear and numerical nonlinear results that they presented both predicted that the free surface would be wave free for supercritical flows and would contain a downstream wavetrain for subcritical flows. No restriction on the dimensionless semicircle radius was found to be necessary for the existence of steady flows and the work appears to be limited purely by the amount of computing time needed to solve the nonlinear integro–differential equations.

Another technique that has been applied to the periodic-waves problem by Bloor (1978) is that of direct conformal transformation of the physical plane onto a half-plane. The transformation used is a generalization of the Schwartz–Christoffel transformation and involves the Hilbert transform of the free-surface angle. As a result of mapping onto a half-plane a simple complex potential can be written and this, together with the free-surface condition, yields a nonlinear integro–differential equations for the free-surface angle. The classical water-wave theories were recovered by expansion of this equations for small free-surface angle and numerical solutions for large-amplitude waves were computed.

As suggested by Bloor (1978) the transformation can easily be modified to cope with obstructions to a free-surface flow and an obstruction in the form of a semi-infinite step on the bottom of a stream is the subject of the rest of this paper. After transformation onto a half-plane, application of the exact free-surface condition results in a nonlinear integro–differential equation for the free-surface angle. The solutions to this equation are dependent on the step height and the upstream Froude number. Linearized solutions, based upon small step height are derived and indicate that the nature of the free surface formed is dependent on whether the upstream flow is subcritical or supercritical. Numerical nonlinear solutions to the integro–differential equation confirm the predictions of the linear theory for small step height but discrepancies in quantities, particularly wavelength and drag, are apparent as the step height increases. The accuracy of the numerical solutions is compared with an asymptotically exact hydraulic theory in the supercritical case and very good agreement is found between the two solutions. In the subcritical case the drag on the step is within the range of values predicted by the approximate theory of Benjamin & Lighthill. Furthermore, the wave amplitude associated with the given drag is in close agreement with that predicted by cnoidal wave theory.

2. Mathematical formulation

The steady two-dimensional free-surface flow of an inviscid, incompressible and irrotational fluid over a semi-infinite rectangular step is considered. Far upstream of the step the flow is uniform with speed c and depth h. A Cartesian coordinate system

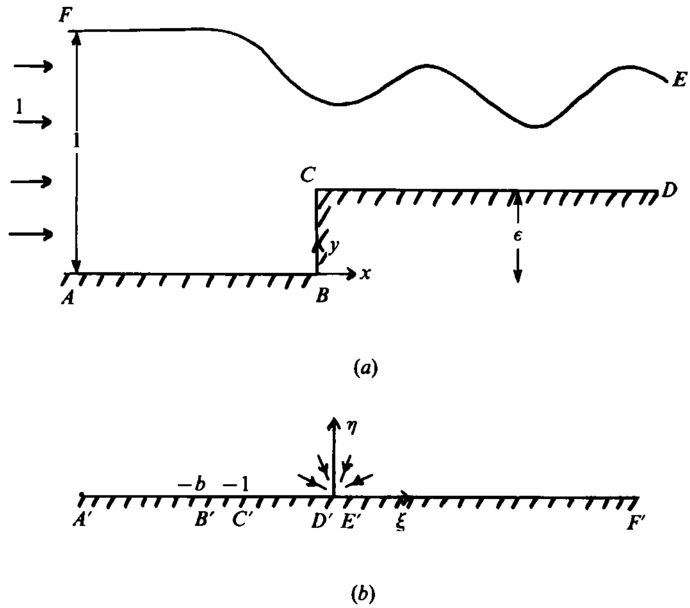

FIGURE 1. The non-dimensional physical plane (*a*) and the corresponding transform plane (*b*).

(X, Y) has its origin at the bottom of the step, which is of height s. The restoring force in the negative Y-direction is gravity. The assumptions made above allow the introduction of a velocity potential Φ and a stream function Ψ. The stream function is chosen so as to have the value zero on the free surface and hence the value $-ch$ along the stepped bottom. The condition on the free surface, where the pressure is uniform, is obtained from Bernoulli's equations. The problem is non-dimensionalized using the transformations

$$x = \frac{X}{h}, \quad y = \frac{Y}{h}, \quad \phi = \frac{\Phi}{ch}, \quad \psi = \frac{\Psi}{ch}, \quad w = \frac{W}{ch}, \tag{2.1}$$

where $w = \phi + \mathrm{i}\psi$ is a complex potential. The Bernoulli condition on the free surface can be written

$$\tfrac{1}{2}F^2q^2 + y_s = \tfrac{1}{2}F^2 + 1, \tag{2.2}$$

where $F = c/(gh)^{\frac{1}{2}}$ is a Froude number, $y_s = y_s(x)$ is the dimensionless free-surface elevation and cq is the fluid speed. The geometry of the flow is shown in figure 1(*a*), where $\epsilon = s/h$ is the dimensionless step height.

The problem is now reduced to finding a complex potential satisfying the appropriate boundary conditions. It is convenient to transform the rather complicated region occupied by the fluid in the physical z-plane into the upper half of the ζ-plane. The transformation used is similar to the one used by Bloor (1978) for the periodic-water-waves problem. With the correspondence of the physical and transformed planes shown in figure 1(*a*) and (*b*) the transformation can be written

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = K(\zeta+b)^{-\frac{1}{2}}(\zeta+1)^{\frac{1}{2}}\zeta^{-1}\exp\left[-\frac{1}{\pi}\int_{t=0}^{\infty}\frac{\theta(t)\,\mathrm{d}t}{\zeta-t}\right], \tag{2.3}$$

where $\zeta = \xi + \mathrm{i}\eta$ and $\theta(t)$ is the angle made with the x-axis by the tangent to the free surface at the point that corresponds to $\xi = t$. When ζ is real and positive the integral in (2.3) becomes a principal value together with a contribution $i\theta(\xi)$ that ensures that $dz/d\zeta$ is an analytic function of ζ.

It is convenient to write

$$P = -\frac{1}{\pi} \fint_{t=0}^{\infty} \frac{\theta(t)}{\xi - t}\, dt,$$

where $\fint$ denotes the Cauchy principal value of the integral. Then on the free surface $\xi > 0$

$$\frac{dz}{d\xi} = K(\xi+b)^{-\frac{1}{2}}(\xi+1)^{\frac{1}{2}}\xi^{-1}\exp\{P + i\theta\}. \tag{2.4}$$

In the uniform flow far upstream of the step the depth of the fluid is unity. Thus by integrating (2.3) along the vertical line from A to F in the z-plane, i.e. from A' to F' around a large semicircle in the ζ-plane, it can be seen that $K = -1/\pi$. In order to relate the height of the step to the location of the point B' in the ζ-plane, (2.3) is integrated over the image of the step in the ζ-plane. This gives

$$\epsilon = -\frac{1}{\pi}\int_{\xi=-b}^{-1} \left|\frac{\xi+1}{\xi+b}\right|^{\frac{1}{2}} \exp\{P\}\frac{d\xi}{\xi}. \tag{2.5}$$

In the ζ-plane the flow is that of a sink of strength $1/\pi$ at the origin giving the complex velocity on the free surface as

$$u - iv = q\,e^{-i\theta} = \left(\frac{\xi+b}{\xi+1}\right)^{\frac{1}{2}} \exp\{-P - i\theta\}, \tag{2.6}$$

where u and v are the x- and y-components of the fluid velocity.

From this stage it will be more convenient to use the variable $\phi = -(1/\pi)\log_e \xi$ rather than ξ, bearing in mind that the complex potential is due to a source of strength $1/\pi$ at the origin in the ζ-plane. Equations (2.4) and (2.6) are now suitable for substitution into the derivative of the Bernoulli equations (2.2). After a little algebra this can be put in the form

$$F^2\left\{\frac{\pi(b-1)\,e^{-\pi\phi}}{(e^{-\pi\phi}+1)^2} - 2\left(\frac{e^{-\pi\phi}+b}{e^{-\pi\phi}+1}\right)\frac{dP}{d\phi}\right\} + 2\left(\frac{e^{-\pi\phi}+1}{e^{-\pi\phi}+b}\right)^{\frac{1}{2}} \exp\{3P\}\sin\theta = 0, \tag{2.7}$$

where

$$P = -\int_{t=-\infty}^{+\infty} \frac{\theta(e^{-\pi t})\, dt}{e^{\pi(t-\phi)} - 1}.$$

Equations (2.5) and (2.7) represent the exact formulation of the problem of free-surface flow over a step in terms of coupled integro-differential equations for the free-surface angle. The solution to these equations is dependent on the two parameters: step height and Froude number. Analytic and numerical solutions to these linearized and nonlinear equations respectively are given later in this paper.

One quantity of physical interest is the drag force on the step caused by the fluid flow. In this idealized model the calculated drag force will represent the change in momentum flux due to the change in the stream produced by the step together with a wave drag in the subcritical case. The drag force D is obtained by integrating the pressure over the step. If the atmospheric pressure is taken to be zero and the hydrostatic component of the drag is subtracted out, it is found that the remaining drag $\rho g h^2 D$ is given by

$$D = \tfrac{1}{2}F^2 \int_{y=0}^{\epsilon} (1 - q^2)\, dy.$$

This can be integrated most conveniently by using (2.4) and (2.6) to express q and y in terms of ϕ to obtain

$$D = \tfrac{1}{2}F^2 \int_{t=-b}^{-1} \left\{1-\left|\frac{t+b}{t+1}\right| \exp\{-2Q\}\right\} \frac{-1}{\pi t} \left|\frac{t+1}{t+b}\right|^{\frac{1}{2}} \exp\{Q\}\,\mathrm{d}t, \tag{2.8}$$

where
$$Q = -\int_{s=-\infty}^{+\infty} \frac{\theta(s)}{t\,\mathrm{e}^{\pi s}-1}\,\mathrm{d}s.$$

3. Approximate methods

Before an attempt is made to solve the full nonlinear equations of §2 it is essential to have some initial estimates of the solution to provide a basis for an iterative numerical solution of the nonlinear equations. This approximate solution to the problem is obtained from a linear theory. Furthermore the parameter range for which solutions are possible in the supercritical case can be determined from a one-dimensional hydraulic theory.

3.1. *Linearized two-dimensional theory*

If the step height ϵ is assumed to be small then the images under the transformation (2.3) of the top and bottom of the step will be close together in the ζ-plane. Writing $b = 1+\delta$, where δ is small, it can be seen that $\theta = O(\delta)$ and (2.7) can be linearized to give

$$\frac{\mathrm{d}P}{\mathrm{d}\phi} - \frac{1}{F^2}\theta = \frac{\delta\pi\,\mathrm{e}^{-\pi\phi}}{2(\mathrm{e}^{-\pi\phi}+1)^2}. \tag{3.1}$$

This equation is readily solved by use of Fourier transforms. The general solution depends upon whether the upstream flow is supercritical ($F > 1$) or subcritical ($F < 1$) and is conveniently written as

$$\theta(\phi) = \frac{\delta F^2}{4\pi} \int_{k=-\infty}^{+\infty} \frac{k\,\mathrm{e}^{-\mathrm{i}k\phi}\,\mathrm{d}k}{\cosh k\{F^2k-\tanh k\}} + H(1-F^2)\{A\cos(k_0\phi)+B\sin(k_0\phi)\}, \tag{3.2}$$

where H is the Heaviside step function, A and B are arbitrary constants and k_0 is the positive root of the equation $F^2k = \tanh k$.

The integral in (3.2) cannot generally be evaluated in closed form. By choosing a suitably deformed contour the integral may be evaluated as an infinite series by using the residue theorem. A full discussion of a similar integral is given by Lamb (1932). For supercritical flow the integral consists of terms that are exponentially small at $\pm\infty$ giving

$$\theta(\phi) = \tfrac{1}{2}\delta F^2 \sum_{n=1}^{\infty} \frac{\beta_n\,\mathrm{e}^{-\beta_n|\phi|}}{\cos\beta_n\{\sec^2\beta_n - F^2\}}, \tag{3.3}$$

where β_n are the positive roots of the equation $F^2\beta = \tan\beta$ $(n = 1, 2, \ldots)$. The special case $F^2 = \infty$, which corresponds to free-streamline flow, gives rise to a closed-form solution of the form

$$\theta(\phi) = \tfrac{1}{4}\delta\,\mathrm{sech}\,(\tfrac{1}{2}\pi\phi). \tag{3.4}$$

For subcritical flow the integral in (3.2) gives rise to wavelike solutions and the condition $\theta(-\infty) = 0$ is satisfied by an appropriate choice of A and B to give

$$\theta(\phi) = \tfrac{1}{2}\delta F^2 \sum_{n=1}^{\infty} \frac{\beta_n\,\mathrm{e}^{-\beta_n|\phi|}}{\cos\beta_n\{\sec^2\beta_n - F^2\}} + \frac{\delta F^2 k_0(\sin(k_0|\phi|)+\sin(k_0\phi))}{2\cosh k_0\{\mathrm{sech}^2 k_0 - F^2\}}, \tag{3.5}$$

where β_n are as defined previously.

The elevation of the free surface is found by integrating (2.4) and then linearizing to obtain

$$\left.\begin{aligned} x(\phi) &= \phi + o(\delta), \\ y(\phi) &= 1 + \int_{s=-\infty}^{\phi} \theta(s)\,\mathrm{d}s + o(\delta). \end{aligned}\right\} \tag{3.6}$$

Linearization of (2.5) gives $\delta = 2\epsilon$ to this order of approximation. The free-surface elevation when $x > 0$ can now be shown to be

$$y_s = 1 + \epsilon F^2 \sum_{n=1}^{\infty} \frac{(2 - \mathrm{e}^{-\beta_n x})}{\cos\beta_n\{\sec^2\beta_n - F^2\}} + o(\epsilon) \qquad (F^2 > 1), \tag{3.7}$$

$$y_s = 1 + \frac{2\epsilon}{\pi}\tan^{-1}[\mathrm{e}^{\pi x/2}] + o(\epsilon) \qquad (F^2 = \infty) \tag{3.8}$$

$$y_s = 1 + \epsilon F^2 \sum_{n=1}^{\infty} \frac{(2 - \mathrm{e}^{-\beta_n x})}{\cos\beta_n\{\sec^2\beta_n - F^2\}} + \frac{2\epsilon F^2(1 - \cos(k_0 x))}{\cosh k_0\{\operatorname{sech}^2 k_0 - F^2\}} + o(\epsilon) \quad (F^2 < 1). \tag{3.9}$$

This linear theory shows that, in supercritical flow, the free surface rises over the step, the slope of the surface becoming more gradual, until far downstream of the step the free surface is asymptotically flat. The depth of fluid far downstream is in general different from that of the fluid upstream and is such that downstream conditions could become critical ($F = 1$) owing to the presence of the step. In the free-streamline case the upstream fluid depths are equal, which is in agreement with accepted theory. For subcritical flows the level of the free surface falls as it approaches the step, while far downstream of the step a periodic wavetrain is predicted. Equation (3.9) shows that the amplitude of these waves depends upon both step height and Froude number whereas the wavelength depends only upon Froude number. This has important ramifications for the numerical solution of the nonlinear equations and is discussed further in the next section. Again it is seen that downstream conditions could become critical due to the change in the mean fluid depth caused by the step.

The drag on the step can be found from (2.8) in the form

$$D = 2\epsilon^2 F^2 \left\{\frac{1}{2\epsilon}\int_{s=-\infty}^{\infty} \frac{\theta(s)\,\mathrm{d}s}{\mathrm{e}^{\pi s}+1} - \frac{1}{4}\right\} + o(\epsilon^2). \tag{3.10}$$

This is evaluated numerically and the results are presented in §5. In the case of free-streamline flow, the integral is elementary giving the drag as zero, as would be expected.

3.2. *One-dimensional theory*

For gravitational flow, the flow far upstream and far downstream of the step is uniform and one-dimensional. The equations of continuity and momentum can therefore be written

$$F^2 + 2 = \frac{F^2}{(y(\infty) - \epsilon)^2} + 2y(\infty), \tag{3.11}$$

$$\frac{F}{F_\infty} = (y(\infty) - \epsilon)^{\frac{3}{2}}, \tag{3.12}$$

where $y(\infty)$ is the elevation of the free surface at infinity and F_∞ is the Froude number based upon the fluid depth and speed at infinity. The condition for (3.11) to have real positive roots is

$$\epsilon \leqslant \tfrac{1}{2}(2 + F^2 - 3F^{\frac{2}{3}}). \tag{3.13}$$

When strict inequality holds the appropriate value of $y(\infty)$ must be taken. If equality holds, (3.11) has a double positive root $y(\infty) = \epsilon + F^{\frac{2}{3}}$. At this step height the downstream Froude number is, by (3.12), equal to one and the flow cannot exist without the presence of a discontinuous hydraulic jump. An expression for the drag on the step can be obtained using a rate-of-change-of-momentum argument. It is found that

$$D = \tfrac{1}{2}\{1 - y^2(\infty) + 2\epsilon(y(\infty) - 1)\} + F^2\left\{1 - \frac{1}{y(\infty) - \epsilon}\right\}. \tag{3.14}$$

The simple theory of this section indicates that solutions to the full two-dimensional problem might be expected for ϵ in the range given by the inequality (3.13). In addition the values predicted by (3.11) and (3.14) will serve as a check on the accuracy of the numerical nonlinear solutions of the next section.

4. Numerical method

To obtain nonlinear solutions to (2.5) and (2.7) it is necessary to resort to a numerical method. A mesh of discrete ϕ- and θ-points is defined by

$$\phi_i = ih, \quad \theta_i = \theta(\phi_i) \quad (-N \leqslant i \leqslant N), \tag{4.1}$$

where h is the discretization interval and N is the number of discrete points either side of the origin. The objective of the numerical method is to find the values of θ_i within this range. Equation (2.7) is decretized, with an error of $O(h^4)$, by using a finite-difference approximation for the derivative to give

$$F^2\left\{\frac{\pi(b-1)\,\mathrm{e}^{-\pi\phi_i}}{(\mathrm{e}^{-\pi\phi_i}+1)^2} - \frac{1}{6h}\left(\frac{\mathrm{e}^{-\pi\phi_i}+b}{\mathrm{e}^{-\pi\phi_i}+1}\right)(-P_{i+2} + 8P_{i+1} - 8P_{i-1} + P_{i-2})\right\}$$
$$+2\left(\frac{\mathrm{e}^{-\pi\phi_i}+1}{\mathrm{e}^{-\pi\phi_i}+b}\right)^{\frac{1}{2}} \exp\{3P_i\} \sin\theta_i = 0 \quad (-N \leqslant i \leqslant N), \tag{4.2}$$

where
$$P_i = -\int_{s=-\infty}^{+\infty} \frac{\theta(s)\,\mathrm{d}s}{\mathrm{e}^{\pi(s-\phi_i)} - 1}.$$

The discretization of P_i is complicated by the singularity in the integrand and the infinite range of integration. The range of integration is truncated to $[\phi_{-2N}, \phi_{2N}]$, the integrand being exponentially small outside this range. The truncated range is subdivided into three subranges, $[\phi_{-2N}, \phi_{i-1}]$, $[\phi_{i-1}, \phi_{i+1}]$ and $[\phi_{i+1}, \phi_{2N}]$. In the first and third of these ranges the integrand is regular and the composite Simpson's rule is used to discretize the integrals. In the range $[\phi_{i-1}, \phi_{i+1}]$ the integrand is singular and the integral is evaluated, in the sense of a Cauchy principal value, by using the Taylor expansion of the integrand about the point ϕ_i. The result, after some algebra, is

$$P_i = -\tfrac{1}{3}h\left[\sum_{j=-2N}^{i-1} \frac{W_j^i\,\theta_j}{\mathrm{e}^{\pi(S_j-\phi_i)} - 1} + \sum_{j=i+1}^{2n} \frac{w_j^i\,\theta_j}{\mathrm{e}^{\pi(S_j-\phi_i)} - 1}\right] + 2h\left[\tfrac{1}{2}\theta_i - \frac{\theta_i'}{\pi}\right] - \tfrac{1}{3}h^3\left[\frac{\theta_i'''}{3\pi} - \tfrac{1}{2}\theta_i'' + \tfrac{1}{6}\pi\theta_i'\right], \tag{4.3}$$

where $S_j = jh$ are the integration points, W_j^i are the weights appropriate to a Simpson's rule and $\theta_i', \theta_i'', \theta_i'''$ are the first three derivatives of θ_i. The truncation error in (4.3) is $O(h^4)$. By replacing the derivatives by appropriate finite-difference representations P_i can be represented as a weighted linear combination of the θ_i, $-2N < i < 2N$ with a truncation error of $O(h^4)$. The values of θ_i in the range $-2N < i < -N$ are taken to be the linear values of §3, i.e. (3.3) and (3.5) for supercritical flows and subcritical flows respectively. The values of θ_i in the range $N < i < 2N$ are again taken from (3.3) for supercritical flows. In subcritical flows these values are found by periodically extending the nonlinear θ_i from the range $-N < i < N$ into the range $N < i < 2N$. This extrapolation is performed to an accuracy of $O(h^4)$. The singularity in (2.5) is removed by some elementary transformations to give, after discretizing by Simpson's rule,

$$\epsilon = \frac{2(1-b)\,h_1}{3\pi} \sum_{j=0}^{2N_1} \alpha_j \cos^2\psi_j \frac{\exp(Q_j)}{[(1-b)\sin^2\psi_j - b]}, \tag{4.4}$$

where $\psi_j = jh_1$ are the integration points, $2N_1 h_1 = \frac{1}{2}\pi$ defines the way the range of integration $[0, \frac{1}{2}\pi]$ is partitioned and α_j are the weights. The value of h_1 was chosen to be as close to h as possible in all calculations. The functional Q_j contains an infinite range of integration which is dealt with in the same manner as P_i to give

$$Q_j = -\tfrac{1}{3}h \sum_{k=-2N}^{2N} \frac{W_k\,\theta_k}{[(b-1)\sin^2\psi_j - b]\,e^{\pi s_k} - 1}. \tag{4.5}$$

Equations (4.2), (4.3), (4.4) and (4.5) are recognized to be a set of $2N+2$ nonlinear algebraic equations for the $2N+2$ unknowns θ_i, $-N < i < N$ and b.

The solution of these equations was carried out by a hybrid Powell's method (Rabinowitz 1970) using the linear solutions of §3 as a first approximation. The method converged quickly to the solution of the nonlinear equations provided the step height was not too large; typically four iterative cycles were necessary to obtain the nonlinear solution. When the step height was large a previously obtained nonlinear solution was used as a first approximation instead of the analytic linear solution. The residual errors in the nonlinear solution, found by substituting a converged solution back into (4.2), (4.3), (4.4) and (4.5) were less in L_2 norm than 10^{-6} for all results presented in this paper.

As a further check on the numerical method a converged solution was substituted into an integrated version of (2.7) and the residual error was found to be of the same order of magnitude in both cases. Various numerical experiments on mesh size and mesh fineness were performed in order to determine when the numerical solution became mesh independent. For subcritical flows with $F = 0.5$ it was found that $N = 80, h = 0.125$, i.e. 161 points representing the free surface, gave satisfactory results whereas for supercritical flows with $F = 2.0$ a mesh with $N = 40, h = 0.2$, i.e. 81 points representing the free surface, was sufficient. Results at these Froude numbers were typical of all results that were obtained and are shown in §5. The free-surface elevation and drag were calculated by numerical integration of (2.4) and (2.8). The singularities in these equations were removed in a similar manner to the one employed in (2.5) prior to numerical integration by Simpson's rule.

5. Numerical results

5.1. *Supercritical flows* ($F^2 > 1$)

Nonlinear solutions for supercritical flows were found throughout a wide range of Froude numbers and step heights. Results showing the free-surface profile and drag at $F = 2.0$, over a range of step heights, are given in figures 2 and 3. These results are typical of the type of solution found and demonstrate the monotonic rise in free-surface elevation with rise to step height. The range of step heights for which solutions could be found at this Froude number was $0 < \epsilon < 0.6188$, which is in good

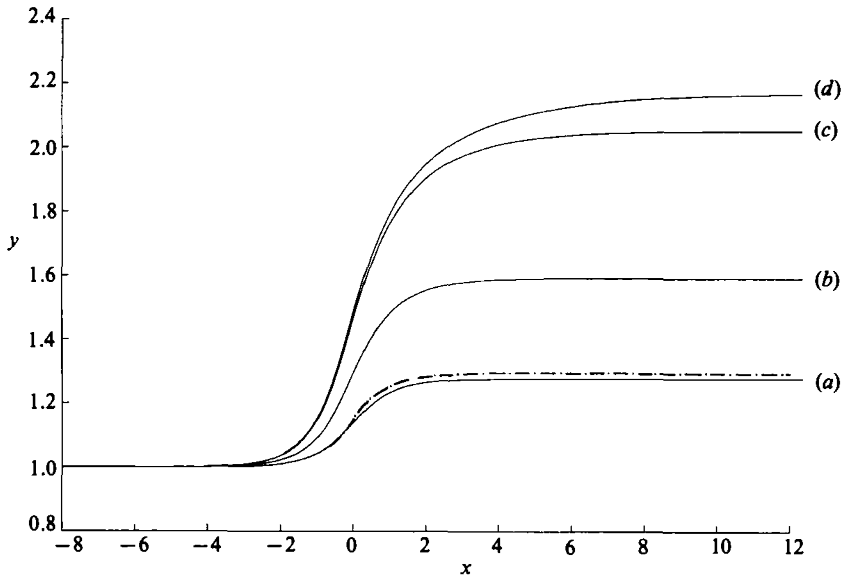

FIGURE 2. Nonlinear solutions at $F = 2$ for various values of ϵ. (a) $\epsilon = 0.2$, (b) 0.4, (c) 0.6, (d) 0.6188. Also shown (–·–) is the linear solution for $\epsilon = 0.2$.

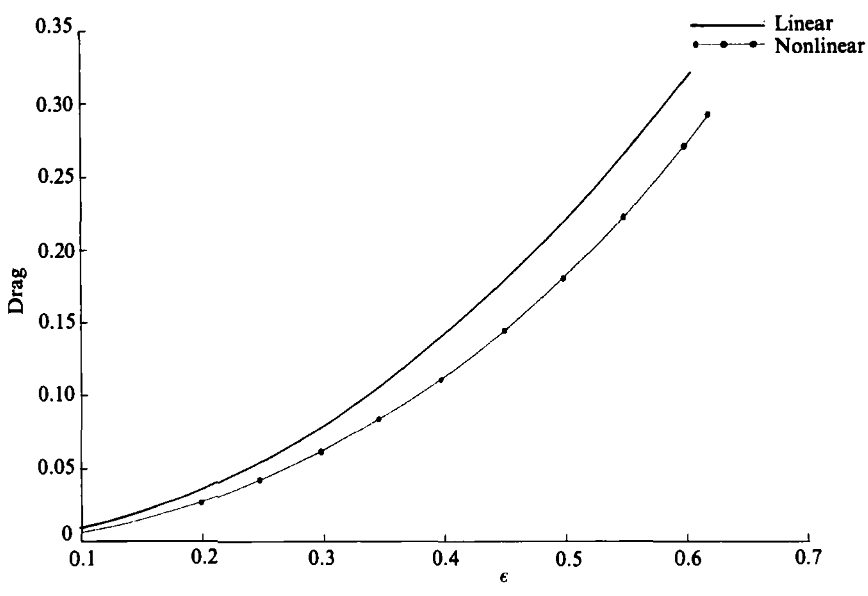

FIGURE 3. The drag on the step as a function of step height compared with the linear solution for supercritical flow.

agreement with the range predicted by inequality (3.13). No solutions were found outside this range.

The difference between the linear and nonlinear solutions only becomes apparent downstream of the step, this difference becoming more apparent as critical downstream conditions are approached. The drag, which is over-estimated by the linear solution, increases monotonically with step height, the relative error of the linear and nonlinear drags being no more than 16 %. As remarked earlier the accuracy of these

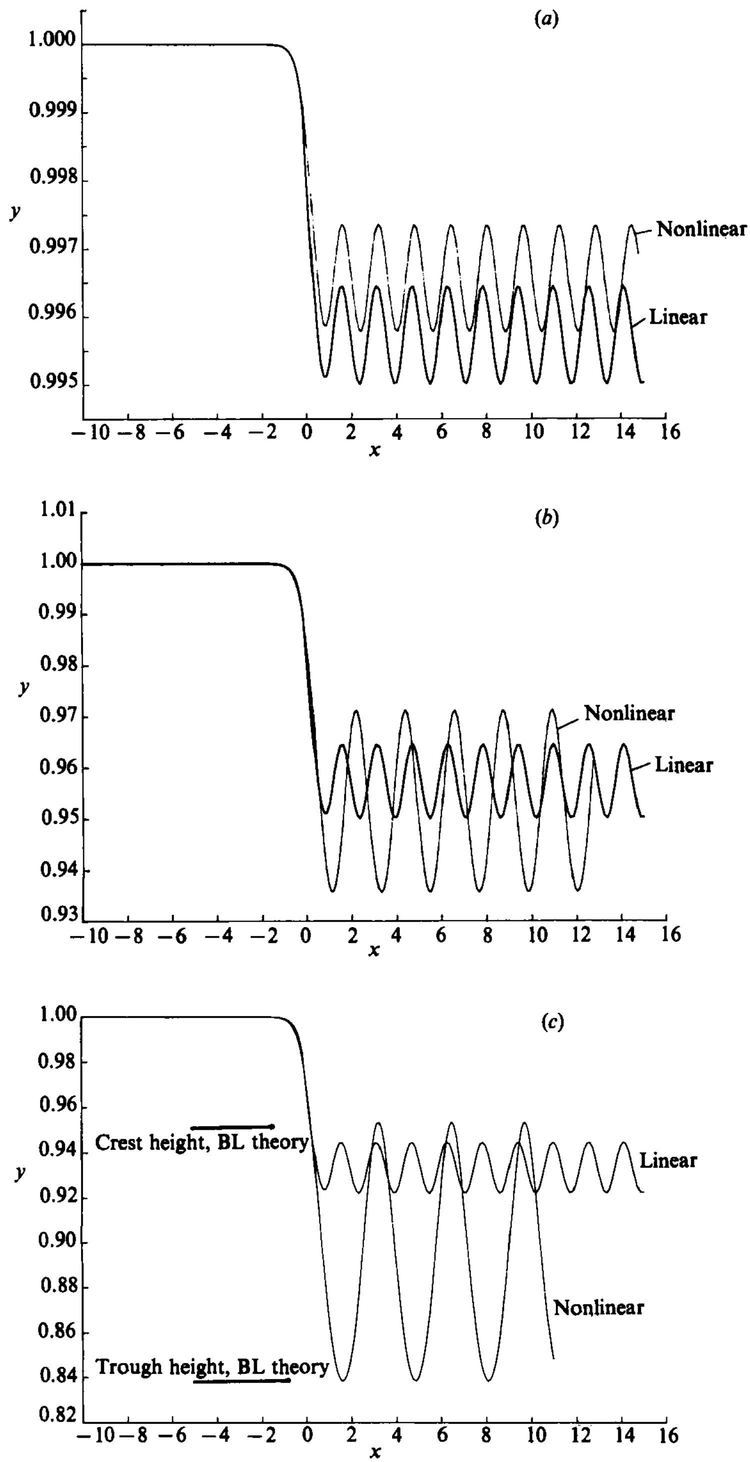

FIGURE 4. The free-surface profile for $F = 0.5$ compared with the linear solution for three values of step height: (*a*) $\epsilon = 0.01$, (*b*) 0.1, (*c*) 0.156, the largest value for which solutions were obtained.

solutions is easily checked by substituting the maximum free-surface elevation and drag into (3.11) and (3.14). For $\epsilon = 0.2$ the numerical method gives a maximum elevation of 1.2744 which is in error by $O(10^{-4})$ while for $\epsilon = 0.6188$ the nonlinear drag was 0.2934, which is in error by $O(10^{-4})$ from the exact result 0.2929. These results are consistent with the discretization error in the numerical modelling of the equations.

Forbes & Schwartz (1982), when discussing supercritical flows over a semicircular obstruction with an otherwise flat bottom, conjectured that the limiting form of the free-surface profile would have a sharp 120° corner at its peak. Clearly a similar conjecture could be made with regard to the flow over a step. No numerical evidence was found that would support this idea, although the present method would fail to converge in such circumstances, because the stagnation point at the corner would imply a singularity in the functional P.

5.2. *Subcritical flow* ($F^2 < 1$)

The range of parameter values for which the nonlinear, wavelike, subcritical flows can be found is restricted computationally by the need to fit several wavelengths of the solution in the positive half of the mesh. The wavelength is determined as part of the iterative solution and was found to become large as the downstream flow became critical. Increasing the mesh size was also unhelpful as this tended to obscure the flow detail above the step. Results showing the free-surface profile and drag at $F = 0.5$ and a range of step heights are given in figures 4 and 5. For very small steps ($\epsilon = 0.01$) the amplitude and wavelength of the linear and nonlinear solutions are in good agreement as shown in figure 4(*a*). As the step height is increased a distinct difference appears between the solutions. At $\epsilon = 0.10$ the amplitude of the nonlinear solution is $O(\epsilon)$ different from the linear amplitude whereas the nonlinear wavelength is $O(1)$ different from the linear wavelength; see figure 4(*b*). For $\epsilon = 0.156$, the largest step for which a solution at this Froude number was obtained, figure 4(*c*) shows the amplitude and wavelength of the linear solution to be grossly in error.

At this stage the nonlinear solution is showing the classical narrow-peak, broad-trough characteristics of the Stokes' theories and values of wave amplitude and wavelength, when adjusted to the local depth of fluid under the wave, are in good agreement with those predicted by these classical theories. It is well known that larger amplitude waves than the ones presented here have been calculated numerically. Cokelet (1977) tabulates a solution for a wave that has a maximum amplitude to wavelength ratio of 0.1436 whereas the largest wave of similar characteristics calculated in this paper had a ratio of 0.062. For flow over a semicircular bump at $F = 0.5$ Forbes & Schwartz were able to generate waves numerically up to a maximum non-dimensional amplitude of ≈ 0.13. At the same Froude number the largest wave presented here has a comparable amplitude of ≈ 0.11. A finer, more extensive mesh would no doubt enable solutions to be found nearer the upper limit of the range predicted by previous numerical work. However, the amount of computing time available precludes this at present.

The drag on the step in subcritical flows is in general negative. This reflects a reduction in the total drag on its hydrostatic level caused by the fall in free-surface elevation above the step. An estimate of the wave drag on the step can be calculated by subtracting from (2.8) the contribution to the drag made by the mean one-dimensional flow. Results for the wave drag are shown in figure 5. These results again demonstrate the importance of nonlinear effects as the step increases in size.

In order to make comparisons with the BL theory, the values of R and S for the

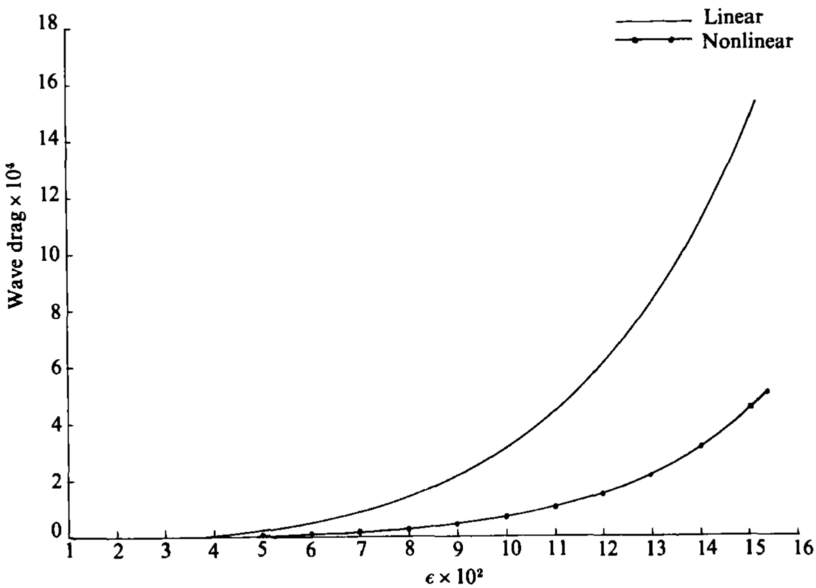

FIGURE 5. The drag on the step due to the presence of the waves as a function of step height for $F = 0.5$. Also shown is the drag given by the linear theory.

present numerical results for a particular value of Q are calculated. It is convenient to non-dimensionalize R and S by R_c and S_c respectively, where the suffix c refers to critical conditions for the particular value of Q. Thus

$$r = \tfrac{1}{3}R/2g^{\frac{2}{3}}Q^{\frac{2}{3}}, \quad s = \tfrac{1}{3}S/2g^{\frac{1}{3}}Q^{\frac{4}{3}}.$$

Now the uniform flow upstream of the step is represented by a point that lies on the upper branch of the cusped curve in figure 6 and for $F = 0.5$ is denoted by the point P_0.

In their analysis, Benjamin & Lighthill considered an obstacle spanning the subcritical stream and, since no dissipation occurred, the drag on the obstacle was represented by a change in s at constant r. Thus all possible wavelike flows downstream of the obstacle could be represented by the points on the line $r =$ constant between the upper and lower branches of the curve in figure 6, and hence an upper bound on the drag was determined.

In interpreting the present results in terms of the BL theory a little care must be taken owing to the change in level of the bottom in this problem. The total head R is measured from the bottom of the channel and hence, since energy is conserved at the step, the value of R downstream of the step is $g\epsilon h$ less than its value upstream. However, Q is clearly the same both upstream and downstream of the step and thus the scales R_c and S_c are also the same. This means that figure 6 can be used to illustrate the present numerical results directly. The point P_0 represents the upstream uniform flow and the points $P_1, P_2, \ldots, P_5$ represents the present numerical wavelike solutions corresponding to $\epsilon = 0.05, 0.1, 0.12, 0.142, 0.156$ respectively. It is clear that the numerical solutions fall in the area of (r, s)-space predicted by the BL theory.

Of course, the BL theory does not actually predict the wave drag and waveform associated with a particular obstacle but it seems worthwhile to compare the approximate cnoidal wave amplitude with the numerical solution. For $\epsilon = 0.156$, the values of r and s can be found from the numerical results and with these values

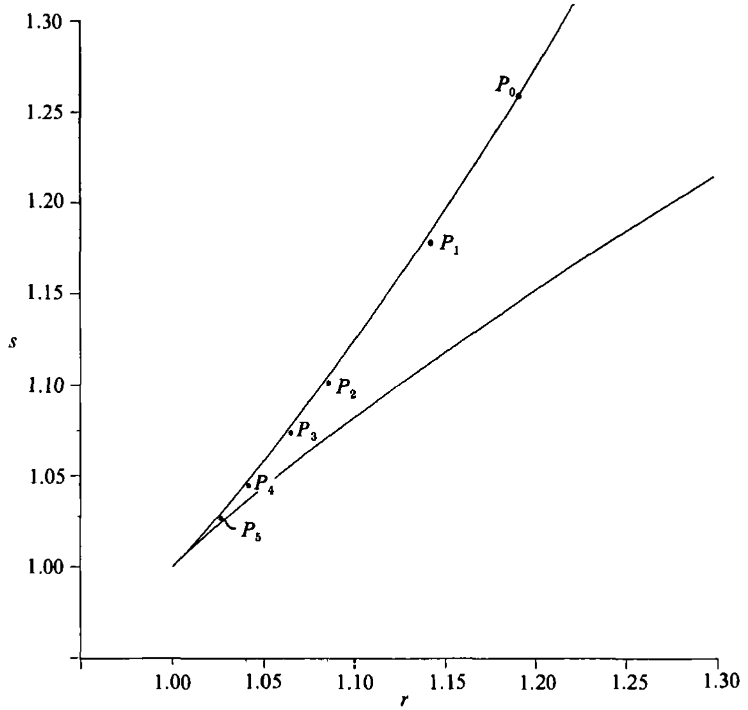

FIGURE 6. Energy *vs.* momentum-flux diagram for flow over a step. P_0 corresponds to uniform subcritical flow, $P_1, P_2, \ldots, P_5$ represent wavelike solutions, all at $F = 0.5$.

the amplitude of the cnoidal wave system downstream is easily predicted from the BL theory. The comparison with the numerical solution is shown in figure 4 (*c*) and it can be seen that the agreement is remarkably good.

It was pointed out in §1 that the method of directed fluid sheets employed by Naghdi & Vonsarnpigoon (1986*a*) reduced to the cnoidal wave theory in the absence of dissipation. In view of this and the comparisons made above it might be thought that good agreement would be obtained with the directly comparable problem of flow over a step. However, with $\epsilon = 0.156$ and an upstream Froude number of 0.5, the theory of Naghdi & Vonsarnpigoon would predict a downstream wavetrain of cnoidal waves of amplitude $0.242h$, which is more than twice what it should be. The reason for this large discrepancy is the boundary condition used at the discontinuity on the downstream height of the wave crest, namely that the crest height is the same as the height of the undisturbed upstream free surface. This essentially fixes the drag on the step. It can be seen from the numerical solution that in fact for this value of ϵ the crest height is about $0.047h$ less than Naghdi & Vonsarnpigoon's value.

6. Conclusions

Steady two-dimensional free-surface flow over a step has been investigated analytically and numerically. The transformation technique used to reduce the physical problem to an integro–differential equations for the free-surface angle θ can, with minor modifications, be used to deal with any polygonal or curved obstruction which could be partially immersed, immersed, or part of the bottom. Both the two-dimensional linear solutions for small step heights and the numerical nonlinear solutions show that there is a rise in free-surface elevation for supercritical ($F^2 > 1$) flows and a depression of the free-surface elevation together with a wavetrain for subcritical flows ($F^2 < 1$). The range of step heights ϵ and Froude numbers F in which solutions for steady supercritical flow over a step exist, determined by one-dimensional theory, coincided with the range in which it was possible to determine numerical solutions to the two-dimensional nonlinear problem.

Reasonable agreement is found between the linear and numerical nonlinear solutions when the step is small. As the step height is increased the supercritical linear flow gradually loses its accuracy whereas the subcritical linear flow quickly becomes totally inaccurate. In particular, the wavelength of the downstream wavetrain is quite different from that predicted by the linear theory.

Previous work on solving free-surface flows over obstructions has also utilized a transformation technique although the transformation used has only relocated the image of the obstruction in a convenient way, leaving the free surface still unknown. The transformation used here relocates the obstruction and the free surface into known positions which enables a complex potential to be written. It is worth pointing out that the singular integral P is evaluated by a special method in §4 and it was the choice of this method that was found to give the most accurate numerical results when solving the nonlinear integro-differential equation.

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